Lecture 31

In this lecture we look at random variables and probability distributions. We introduce the binomial probability distribution.

Random Variables

Let S be a sample space. A <u>random variable</u> on S is a variable X for which each possible value x is a real number associated with a particular outcome from the sample space.

Normally X ranges over some discrete subset of $\mathbb{R}$, such as the set $\{0, 1, 2, \ldots, n\}$ for some positive integer n.

<u>E.g.</u> Suppose we toss a coin. The random variable X could mean "the number of heads". So when a head shows up we have $X = 1$ and when a tail appears we have $X = 0$.

<u>E.g.</u> Returning to the coin-tossing event, we could regard a "head" as Outcome 1 and a "tail" as Outcome 2. Then the random variable X could be defined to be $X=1$ when a "head" appears and $X=2$ when a tail appears.

<u>E.g.</u> Suppose we cast a die. The elementary outcomes already form a finite set of integers $\{1,2,3,4,5,6\}$, so it's a very simple matter to just let X be the score.

<u>E.g.</u> Returning to the last example, we could instead let X be the number of successes, where "success" means getting 5 or 6.

Then $X = 0$ if the die shows 1, 2, 3 or 4, and $X = 1$ if the die shows 5 or 6.

Note that we can assign probabilities to the values of X. In the last example, $Pr(X=0) = Pr(\{1,2,3,4\}) = 4/6 = 2/3$ while $Pr(X=1) = Pr(\{5,6\}) = 2/6 = 1/3$.

<u>Probability Distributions</u>

Given a random variable X on a sample space S, the <u>probability distribution function</u> is the function defined by

$$f(x) = Pr(X = x).$$

The <u>cumulative probability distribution function</u> is the function defined by

$$F(x) = Pr(X \leq x).$$

A "probability distribution" refers to the way that the values of a random variable are distributed, as specified by the probability distribution function.

E.g. The following table describes $f(x)$ and $F(x)$ when X = "number of heads" if a coin is tossed. For convenience the values of X are listed in increasing order.

Outcome	X	$Pr(X=x)$	$Pr(X \le x)$
T	0	$1/2$	$1/2$
H	1	$1/2$	1

E.g. Now we show the table for a die-casting event when X is the score.

Outcome	X	$Pr(X=x)$	$Pr(X \le x)$
1	1	$1/6$	$1/6$
2	2	$1/6$	$2/6 = 1/3$
3	3	$1/6$	$3/6 = 1/2$
4	4	$1/6$	$4/6 = 2/3$
5	5	$1/6$	$5/6$
6	6	$1/6$	1

Notice that a probability distribution function has to have the property that all probabilities add up to 1.

<u>E.g.</u> If we regard 5 or 6 as "success" and < 5 as "failure" when a die is cast, then the table looks like this.

Outcome	X	$Pr(X=x)$	$Pr(X \le x)$
1, 2, 3, 4	0	$2/3$	$2/3$
5, 6	1	$1/3$	1

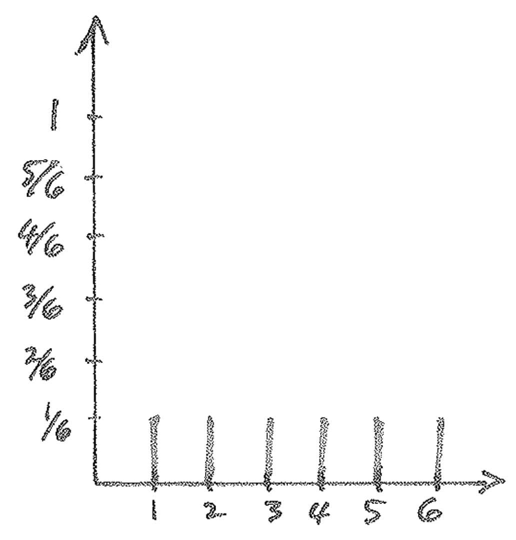
Special Kinds of Probability Distributions

Finite Uniform Distribution

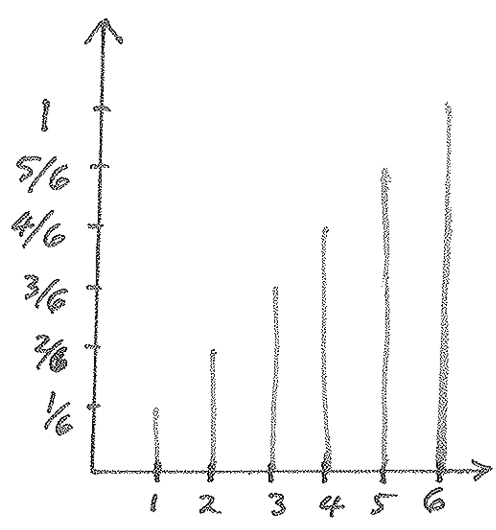
In this distribution X takes values $1, 2, \ldots, n$ with all values being equiprobable.

E.g.

A previous example had X taking values 1 to 6 when a die is cast. Here are the graphs of $f(x)$ and $F(x)$.



Prob. dist.



Cum. dist.

Binomial Distribution

Suppose that we have a sample space S and we classify some outcomes as representing "success" while the others represent "failure". We let

$$p = \Pr(\text{success})$$

and $\quad q = 1-p = \Pr(\text{failure}).$

Let the random variable X be the number of successes when the experiment associated with the sample space is carried out n times. So X ranges from 0 to n. Then the probability of k success is

$$\Pr(X=k) = {}^{n}C_{k}\, p^{k}(1-p)^{n-k} .$$

The probability distribution represented by this function is called the <u>binomial distribution.</u>

The factor ^nC_k is present because it gives the number of ways that the k successes may be distributed among the n iterations of the experiment. Because the repeated trials are independent of each other, the probability of k successes and $(n-k)$ failures in specified positions among the n trials is just the product of all the individual probabilities, which gives us the factor $p^k(1-p)^{n-k}$.

In a binomial distribution the probability of k successes is sometimes denoted by $B_{n,p}(k)$, so that

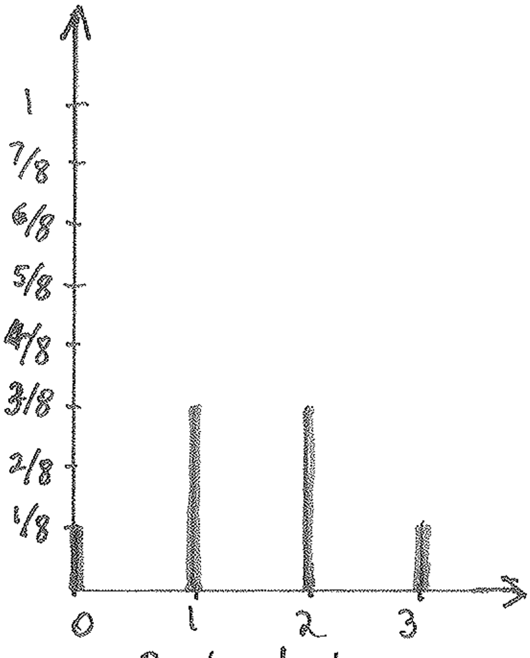
$$B_{n,p}(k) = {}^nC_k \, p^k q^{n-k} \quad .$$

E.g. A coin is tossed 3 times. What are the probabilities of 0, 1, 2 or 3 heads?

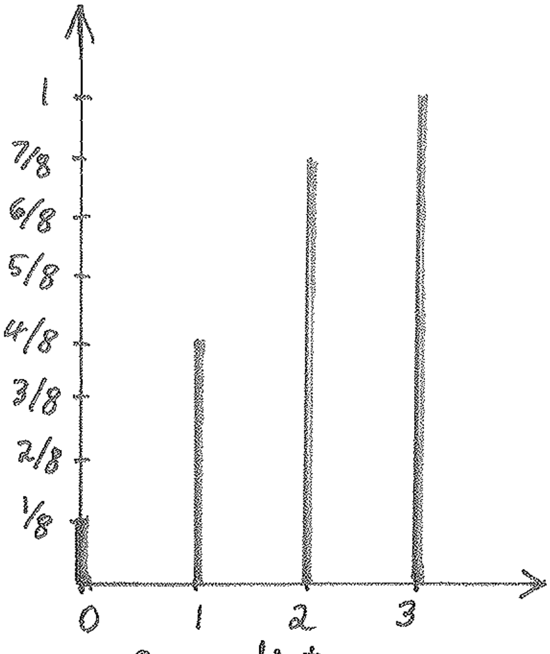
Sol$^{\underline{n}}$ Let X be the number of heads.

Then
$$Pr(X=0) = {}^3C_0\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8},$$
$$Pr(X=1) = {}^3C_1\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^2 = \tfrac{3}{8},$$
$$Pr(X=2) = {}^3C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^1 = \tfrac{3}{8}, \quad \text{and}$$
$$Pr(X=3) = {}^3C_3\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^0 = \tfrac{1}{8}.$$

Here are the graphs of $f(x)$ and $F(x)$.



Prob. dist.



Cum. dist.